Asymptotics for the survival probability in a killed branching random walk

by

Nina Gantert, Yueyun Hu and Zhan Shi

Universität Münster, Université Paris XIII & Université Paris VI

`This version: February 15, 2010`

Summary. Consider a discrete-time one-dimensional supercritical branching random walk. We study the probability that there exists an infinite ray in the branching random walk that always lies above the line of slope $\gamma - \varepsilon$, where γ denotes the asymptotic speed of the right-most position in the branching random walk. Under mild general assumptions upon the distribution of the branching random walk, we prove that when $\varepsilon \to 0$, this probability decays like $\exp\{-\frac{\beta+o(1)}{\varepsilon^{1/2}}\}$, where β is a positive constant depending on the distribution of the branching random walk. In the special case of i.i.d. Bernoulli(p) random variables (with $0 < p < \frac{1}{2}$) assigned on a rooted binary tree, this answers an open question of Robin Pemantle, see [19].

Keywords. Branching random walk, survival probability, maximal displacement.

2000 Mathematics Subject Classification. 60J80.

1 Introduction

We consider a one-dimensional branching random walk in discrete time. Before introducing the model and the problem, we start with an example, borrowed from Pemantle [19], in the study of binary search trees.

Example 1.1 Let $\mathbb{T}_{\mathtt{bs}}$ be a binary tree ("$\mathtt{bs}$" for binary search), rooted at e. Let $(Y(x),\, x \in \mathbb{T}_{\mathtt{bs}})$ be a collection, indexed by the vertices of the tree, of i.i.d. Bernoulli random variables with mean $p \in (0,\, \frac{1}{2})$. For any vertex $x \in \mathbb{T}_{\mathtt{bs}} \backslash \{e\}$, let $[\![e,\, x]\!]$ denote the shortest path connecting e with x, and let $]\!]e,\, x]\!] := [\![e,\, x]\!] \backslash \{e\}$. We define

$$U_{\mathtt{bs}}(x) := \sum_{v \in \,]\!]e,\, x]\!]} Y(v), \qquad x \in \mathbb{T}_{\mathtt{bs}} \backslash \{e\},$$

and $U_{\mathtt{bs}}(e) := 0$. Then $(U_{\mathtt{bs}}(x),\, x \in \mathbb{T}_{\mathtt{bs}})$ is a binary branching Bernoulli random walk. It is known (Kingman [14], Hammersley [8], Biggins [2]) that

$$\lim_{n\to\infty} \frac{1}{n} \max_{|x|=n} U_{\mathtt{bs}}(x) = \gamma_{\mathtt{bs}}, \qquad \text{a.s.},$$

where the constant $\gamma_{\mathtt{bs}} = \gamma_{\mathtt{bs}}(p) \in (0,\, 1)$ is the unique solution of

$$\gamma_{\mathtt{bs}} \log \frac{\gamma_{\mathtt{bs}}}{p} + (1-\gamma_{\mathtt{bs}}) \log \frac{1-\gamma_{\mathtt{bs}}}{1-p} - \log 2 = 0. \tag{1.1}$$

For any $\varepsilon > 0$, let $\varrho_{\mathtt{bs}}(\varepsilon, p)$ denote the probability that there exists an infinite ray[1] $\{e =: x_0,\, x_1,\, x_2,\, \ldots\}$ such that $U_{\mathtt{bs}}(x_j) \ge (\gamma_{\mathtt{bs}} - \varepsilon) j$ for all $j \ge 1$. It is conjectured by Pemantle [19] that there exists a constant $\beta_{\mathtt{bs}}(p)$ such that[2]

$$\log \varrho_{\mathtt{bs}}(\varepsilon, p) \;\sim\; -\frac{\beta_{\mathtt{bs}}(p)}{\varepsilon^{1/2}}, \qquad \varepsilon \to 0. \tag{1.2}$$

We prove the conjecture, and give the value of $\beta_{\mathtt{bs}}(p)$. Let $\psi_{\mathtt{bs}}(t) := \log[2(pe^t + 1 - p)]$, $t > 0$. Let $t^* = t^*(p) > 0$ be the unique solution of $\psi_{\mathtt{bs}}(t^*) = t^* \psi'_{\mathtt{bs}}(t^*)$. [One can then check that the solution of equation (1.1) is $\gamma_{\mathtt{bs}} = \frac{\psi_{\mathtt{bs}}(t^*)}{t^*}$.] Our main result, Theorem 1.2 below, implies that conjecture (1.2) holds, with

$$\beta_{\mathtt{bs}}(p) := \frac{\pi}{2^{1/2}} [t^* \psi''_{\mathtt{bs}}(t^*)]^{1/2}.$$

A particular value of $\beta_{\mathtt{bs}}$ is as follows: if $0 < p_0 < \frac{1}{2}$ is such that $16 p_0 (1-p_0) = 1$ (i.e., if $\gamma_{\mathtt{bs}}(p_0) = \frac{1}{2}$), then

$$\beta_{\mathtt{bs}}(p_0) = \frac{\pi}{4} \Big(\frac{\gamma'_{\mathtt{bs}}(p_0)}{1-2p_0} \Big)^{1/2} \log \frac{1}{4p_0},$$

[1] By an infinite ray, we mean that each x_j is the parent of x_{j+1}.

[2] Throughout the paper, by $a(\varepsilon) \sim b(\varepsilon)$, $\varepsilon \to 0$, we mean $\lim_{\varepsilon \to 0} \frac{a(\varepsilon)}{b(\varepsilon)} = 1$.

where $\gamma'_{\mathtt{bs}}(p_0)$ denotes the derivative of $p \mapsto \gamma_{\mathtt{bs}}(p)$ at p_0. This is, informally, in agreement with the following theorem of Aldous ([1], Theorem 6): if $p \in (p_0, \frac{1}{2})$ is such that $\gamma_{\mathtt{bs}}(p) = \frac{1}{2} + \varepsilon$, then the probability that there exists an infinite ray x with $U_{\mathtt{bs}}(x_i) \ge \frac{1}{2}\, i$, $\forall i \ge 1$, is

$$\exp\Big(- \frac{\pi \log(1/(4p_0))}{4(1-2p_0)^{1/2}} \, \frac{1}{(p-p_0)^{1/2}} + O(1)\Big), \qquad \varepsilon \to 0. \qquad \square$$

As a matter of fact, the main result of this paper (Theorem 1.2 below) is valid for more general branching random walks: the tree $\mathbb{T}_{\mathtt{bs}}$ can be random (Galton–Watson), the random variables assigned on the vertices of the tree are not necessarily Bernoulli, nor necessarily identically distributed, nor necessarily independent if the vertices share a common parent.

Our model is as follows, which is a one-dimensional discrete-time branching random walk. At the beginning, there is a single particle located at position $x = 0$. Its children, who form the first generation, are positioned according to a certain point process. Each of the particles in the first generation gives birth to new particles that are positioned (with respect to their birth places) according to the same point process; they form the second generation. The system goes on according to the same mechanism. We assume that for any n, each particle at generation n produces new particles independently of each other and of everything up to the n-th generation.

We denote by $(U(x),\, |x| = n)$ the positions of the particles in the n-th generation, and by $Z_n := \sum_{|x|=n} 1$ the number of particles in the n-th generation. Clearly, $(Z_n,\, n \ge 0)$ forms a Galton–Watson process. [In Example 1.1, $Z_n = 2^n$, whereas $(U(x),\, |x| = 1)$ is a pair of independent Bernoulli(p) random variables.]

We assume that for some $\delta > 0$,

$$\mathbf{E}(Z_1^{1+\delta}) < \infty, \qquad \mathbf{E}(Z_1) > 1; \tag{1.3}$$

in particular, the Galton–Watson process $(Z_n,\, n \ge 0)$ is supercritical. We also assume that there exist $\delta_+ > 0$ and $\delta_- > 0$ such that

$$\mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{\delta_+ U(x)} \Big) < \infty, \qquad \mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{-\delta_- U(x)} \Big) < \infty. \tag{1.4}$$

An additional assumption is needed (which in Example 1.1 corresponds to the condition $p < \frac{1}{2}$). Let us define the logarithmic generating function for the branching walk:

$$\psi(t) := \log \mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{tU(x)} \Big), \qquad t > 0. \tag{1.5}$$

Let $\zeta := \sup\{t : \psi(t) < \infty\}$. Under Condition (1.4), we have $0 < \zeta \le \infty$, and ψ is C^∞ on $(0, \zeta)$. We assume that there exists $t^* \in (0, \zeta)$ such that

$$\psi(t^*) = t^* \psi'(t^*). \tag{1.6}$$

For discussions on this condition, see the examples presented after Theorem 1.2 below.

Recall that (Kingman [14], Hammersley [8], Biggins [2]) conditioned on the survival of the system,

$$\lim_{n\to\infty} \frac{1}{n} \max_{|x|=n} U(x) = \gamma, \qquad \text{a.s.,} \tag{1.7}$$

where $\gamma := \frac{\psi(t^*)}{t^*}$ is a constant, with t^* and $\psi(\cdot)$ defined in (1.6) and (1.5), respectively.

For $\varepsilon > 0$, let $\varrho_U(\varepsilon)$ denote the probability that there exists an infinite ray $\{e =: x_0, x_1, x_2, \ldots\}$ such that $U(x_j) \ge (\gamma - \varepsilon) j$ for all $j \ge 1$. Our main result is as follows.

Theorem 1.2 *Assume (1.3) and (1.4). If (1.6) holds, then*

$$\log \varrho_U(\varepsilon) \;\sim\; -\frac{\pi}{(2\varepsilon)^{1/2}} \, [t^* \psi''(t^*)]^{1/2}, \qquad \varepsilon \to 0, \tag{1.8}$$

where t^* and ψ are as in (1.6) and (1.5), respectively.

Since $(U(x),\ |x| = 1)$ is not a deterministic set (excluded by the combination of (1.6) and (1.3)), the function ψ is strictly convex on $(0, \zeta)$. In particular, we have $0 < \psi''(t^*) < \infty$.

We now present a few simple examples to illustrate the meaning of Assumption (1.6). For more detailed discussions, see Jaffuel [11].

Example 1.1 (continuation). In Example 1.1, Conditions (1.3) and (1.4) are obviously satisfied, whereas (1.6) is equivalent to $p < \frac{1}{2}$. In this case, (1.8) becomes (1.2). Clearly, if $p > \frac{1}{2}$, $\varrho_{\mathtt{bs}}(\varepsilon, p)$ does not go to 0 because the vertices labeled with 1 percolate, with positive probability, on the tree. $\square$

Example 1.3 Consider the example of Bernoulli branching random walk, i.e., such that $U(x) \in \{0, 1\}$ for any $|x| = 1$; to avoid trivial cases, we assume $\mathbf{E}(\sum_{|x|=1} \mathbf{1}_{\{U(x)=1\}}) > 0$ and $\mathbf{E}(\sum_{|x|=1} \mathbf{1}_{\{U(x)=0\}}) > 0$.

Condition (1.4) is automatically satisfied as long as we assume (1.3). Elementary computations show that Condition (1.6) is equivalent to $\mathbf{E}(\sum_{|x|=1} \mathbf{1}_{\{U(x)=1\}}) < 1$. (In particular, if we assign independent Bernoulli(p) random variables on the vertices of a rooted binary tree, we recover Example 1.1). Again, if $\mathbf{E}(\sum_{|x|=1} \mathbf{1}_{\{U(x)=1\}}) > 1$, $\varrho_U(\varepsilon)$ does not go to 0 because the vertices labeled with 1 percolate, with positive probability, on the tree. $\square$

Example 1.4 Assume the distribution of U is bounded from above, in the sense that there exists a constant $C \in \mathbb{R}$ such that $\sup_{|x|=1} U(x) \le C$. Let $s_U := \operatorname{ess\,sup} \sup_{|x|=1} U(x) = \sup\{a \in \mathbb{R} : \mathbf{P}\{\sup_{|x|=1} U(x) \ge a\} > 0\} < \infty$. Under (1.3) and (1.4), Condition (1.6) is satisfied if and only if $\mathbf{E}(\sum_{|x|=1} \mathbf{1}_{\{U(x)=s_U\}}) < 1$. □

Example 1.5 Assume that (1.3) holds true. If $\operatorname{ess\,sup} \sup_{|x|=1} U(x) = \infty$, then Condition (1.6) is satisfied. □

We mention that the question we address here in the discrete case has a continuous counterpart, which has been investigated in the context of the F-KPP equation with cutoff, see [5], [6], [7].

The rest of the paper is organized as follows. In Section 2, we make a linear transformation of our branching random walk so that it will become a boundary case in the sense of Biggins and Kyprianou [3]; the linear transformation is possible due to Assumption (1.6). Section 3 is devoted to the proof of the upper bound in Theorem 1.2, whereas the proof of the lower bound is in Section 4.

2 A linear transformation

We define

$$V(x) := -t^* U(x) + \psi(t^*)\, |x|. \tag{2.1}$$

Then

$$\mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{-V(x)} \Big) = 1, \qquad \mathbf{E}\Big(\sum_{|x|=1} V(x) \mathrm{e}^{-V(x)} \Big) = 0. \tag{2.2}$$

Since $t^* < \zeta$, there exists $\delta_1 > 0$ such that

$$\mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{-(1+\delta_1) V(x)} \Big) < \infty. \tag{2.3}$$

On the other hand, by (1.4), there exists $\delta_2 > 0$ such that

$$\mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{\delta_2 V(x)} \Big) < \infty. \tag{2.4}$$

The new branching random walk $(V(x))$ satisfies $\lim_{n\to\infty} \frac{1}{n} \inf_{|x|=n} V(x) = 0$ a.s. conditioned on non-extinction. Let

$$\varrho(\varepsilon) = \varrho(V, \varepsilon) := \mathbf{P}\Big\{ \exists \text{ infinite ray } \{e =: x_0,\, x_1,\, x_2,\, \ldots\}\colon\, V(x_j) \le \varepsilon j,\, \forall j \ge 1 \Big\}. \tag{2.5}$$

Theorem 1.2 will be a consequence of the following estimate: assuming (2.2), then

$$\log \varrho(\varepsilon) \sim -\frac{\pi\sigma}{(2\varepsilon)^{1/2}}, \qquad \varepsilon \to 0, \tag{2.6}$$

where σ is the constant in (2.7) below.

It is (2.6) we are going to prove: an upper bound is proved in Section 3, and a lower bound in Section 4.

We conclude this section with a change-of-probabilities formula, which is the raison d'être of the linear transformation. Let $S_0 := 0$, and let $(S_i - S_{i-1},\ i \ge 1)$ be a sequence of i.i.d. random variables such that for any measurable function $f : \mathbb{R} \to [0, \infty)$,

$$\mathbf{E}(f(S_1)) = \mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{-V(x)} f(V(x)) \Big).$$

In particular, $\mathbf{E}(S_1) = 0$ (by (2.2)). In words, (S_n) is a mean-zero random walk. We denote

$$\sigma^2 := \mathbf{E}(S_1^2) = \mathbf{E}\Big(\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)} \Big) = (t^*)^2 \psi''(t^*). \tag{2.7}$$

Since $\mathbf{E}(Z_1^{1+\delta}) < \infty$ (Condition (1.3)) and $\mathbf{E}(\sum_{|x|=1} \mathrm{e}^{-(1+\delta_1)V(x)}) < \infty$ (see (2.3)), there exists $\delta_3 > 0$ such that $\mathbf{E}(\mathrm{e}^{uS_1}) < \infty$ for all $|u| \le \delta_3$.

In view of (2.2), we have, according to Biggins and Kyprianou [3], for any $n \ge 1$ and any measurable function $F : \mathbb{R}^n \to [0, \infty)$,

$$\mathbf{E}\Big(\sum_{|x|=n} \mathrm{e}^{-V(x)} F(V(x_i),\ 1 \le i \le n) \Big) = \mathbf{E}[F(S_i,\ 1 \le i \le n)], \tag{2.8}$$

where, for any x with $|x| = n$, $\{e =: x_0,\ x_1,\ \dots, x_n := x\}$ is the shortest path connecting e to x.

We now give a bivariate version of (2.8). For any vertex x, the number of its children is denoted by $\nu(x)$. Condition (1.3) guarantees that $\mathbf{P}\{\nu(x) < \infty,\ \forall x\} = 1$. In light of (2.2), we have, for any $n \ge 1$ and any measurable function $F : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$,

$$\mathbf{E}\Big(\sum_{|x|=n} \mathrm{e}^{-V(x)} F[V(x_i),\ \nu(x_{i-1}),\ 1 \le i \le n] \Big) = \mathbf{E}\Big(F[S_i,\ \nu_{i-1},\ 1 \le i \le n] \Big), \tag{2.9}$$

where $(S_i - S_{i-1},\ \nu_{i-1})$, for $i \ge 1$, are i.i.d. random vectors, whose common distribution is determined by (recalling that $Z_1 := \#\{y :\ |y| = 1\}$)

$$\mathbf{E}[f(S_1,\ \nu_0)] = \mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{-V(x)} f(V(x),\ Z_1) \Big), \tag{2.10}$$

for any measurable function $f : \mathbb{R}^2 \to [0, \infty)$.

The proof of (2.9), just as the proof of (2.8) in Biggins and Kyprianou [3], relies on a simple argument by induction on n. We feel free to omit it.

[We mention that (2.9) is a special case of the so-called spinal decomposition for branching random walks, a powerful tool developed by Lyons, Pemantle and Peres [16] and Lyons [15]. The idea of spinal decomposition, which goes back at least to Kahane and Peyrière [12], has been used in the literature by many authors in several different forms.]

We now extend a useful result of Mogulskii [18] to *arrays* of random variables.

Lemma 2.1 (A triangular version of Mogulskii [18]) *For each $n \ge 1$, let $X_i^{(n)}$, $1 \le i \le n$, be i.i.d. real-valued random variables. Let $g_1 < g_2$ be continuous functions on $[0, 1]$ with $g_1(0) < 0 < g_2(0)$. Let (a_n) be a sequence of positive numbers such that $a_n \to \infty$ and that $\frac{a_n^2}{n} \to 0$. Assume that there exist constants $\eta > 0$ and $\sigma^2 > 0$ such that*

$$\sup_{n\ge 1} \mathbf{E}(|X_1^{(n)}|^{2+\eta}) < \infty, \quad \mathbf{E}(X_1^{(n)}) = o\Big(\frac{a_n}{n}\Big), \quad \mathrm{Var}(X_1^{(n)}) \to \sigma^2. \tag{2.11}$$

Consider the measurable event

$$E_n := \Big\{ g_1\Big(\frac{i}{n}\Big) \le \frac{S_i^{(n)}}{a_n} \le g_2\Big(\frac{i}{n}\Big), \ \textit{for } 1 \le i \le n \Big\},$$

where $S_i^{(n)} := X_1^{(n)} + \cdots + X_i^{(n)}$. We have

$$\lim_{n\to\infty} \frac{a_n^2}{n} \log \mathbf{P}\{E_n\} = -\frac{\pi^2\sigma^2}{2} \int_0^1 \frac{\mathrm{d}t}{[g_2(t) - g_1(t)]^2}. \tag{2.12}$$

Moreover, for any $b > 0$,

$$\lim_{n\to\infty} \frac{a_n^2}{n} \log \mathbf{P}\Big\{ E_n, \ \frac{S_n^{(n)}}{a_n} \ge g_2(1) - b \Big\} = -\frac{\pi^2\sigma^2}{2} \int_0^1 \frac{\mathrm{d}t}{[g_2(t) - g_1(t)]^2}. \tag{2.13}$$

If the distribution of $X_1^{(n)}$ does not depend on n, Lemma 2.1 is Mogulskii [18]'s result. In this case, condition (2.11) is satisfied as long as $X_1^{(n)}$ is centered, having a finite $(2+\eta)$-moment (for some $\eta > 0$), and such that it is not identically zero.[3]

The proof of Lemma 2.1 is in the same spirit (but with some additional technical difficulties) as in the original work of Mogulskii [18], and is included as an appendix at the end of the paper. We mention that as in [18], it is possible to have a version of Lemma 2.1 when $X_1^{(n)}$ belongs to the domain of attraction of a stable non-Gaussian law, except that the constant $\frac{\pi^2}{2}$ in (2.12)–(2.13) will be implicit.

[3] In this case, we even can allow $\eta = 0$; see [18].

3 Proof of Theorem 1.2: the upper bound

In this section, we prove the upper bound in (2.6):

$$\limsup_{\varepsilon\to 0} \varepsilon^{1/2} \log \varrho(\varepsilon) \le -\frac{\pi\sigma}{2^{1/2}}\,, \tag{3.1}$$

where $\varrho(\varepsilon)$ is defined in (2.5), and σ is the constant in (2.7).

The main idea in this section is borrowed from Kesten [13]. We start with the trivial inequality that for any $n \ge 1$ (an appropriate value for $n = n(\varepsilon)$ will be chosen later on),

$$\varrho(\varepsilon) \le \mathbf{P}\Big\{ \exists\, x:\ |x| = n,\ V(x_i) \le \varepsilon i,\ \forall i \le n \Big\}.$$

Let $(b_i,\, i \ge 0)$ be a sequence of non-negative real numbers whose value (depending on n) will be given later on. For any x, let $H(x) := \inf\{i:\ 1 \le i \le |x|,\, V(x_i) \le \varepsilon i - b_i\}$, with $\inf \emptyset := \infty$. Then $\mathbf{P}\{H(x) = \infty\} + \mathbf{P}\{H(x) \le |x|\} = 1$. Therefore,

$$\varrho(\varepsilon) \le \varrho_1(\varepsilon) + \varrho_2(\varepsilon),$$

where

$$\begin{aligned} \varrho_1(\varepsilon) = \varrho_1(\varepsilon, n) &:= \mathbf{P}\Big\{ \exists\, |x| = n:\ H(x) = \infty,\ V(x_i) \le \varepsilon i,\ \forall i \le n \Big\}, \\ \varrho_2(\varepsilon) = \varrho_2(\varepsilon, n) &:= \mathbf{P}\Big\{ \exists\, |x| = n:\ H(x) \le n,\ V(x_i) \le \varepsilon i,\ \forall i \le n \Big\}. \end{aligned}$$

We now estimate $\varrho_1(\varepsilon)$ and $\varrho_2(\varepsilon)$ separately.

By definition,

$$\begin{aligned} \varrho_1(\varepsilon) &= \mathbf{P}\Big\{ \exists\, |x| = n:\ \varepsilon i - b_i < V(x_i) \le \varepsilon i,\ \forall i \le n \Big\} \\ &= \mathbf{P}\Big\{ \sum_{|x|=n} \mathbf{1}_{\{\varepsilon i - b_i < V(x_i) \le \varepsilon i,\ \forall i \le n\}} \ge 1 \Big\} \\ &\le \mathbf{E}\Big(\sum_{|x|=n} \mathbf{1}_{\{\varepsilon i - b_i < V(x_i) \le \varepsilon i,\ \forall i \le n\}} \Big), \end{aligned}$$

the last inequality being a consequence of Chebyshev's inequality. Applying the change-of-probabilities formula (2.8) to $F(z) := \mathrm{e}^{z_n} \mathbf{1}_{\{\varepsilon i - b_i < z_i \le \varepsilon i,\ \forall i \le n\}}$ for $z := (z_1, \ldots, z_n) \in \mathbb{R}^n$, this yields, in the notation of (2.8),

$$\varrho_1(\varepsilon) \le \mathbf{E}\Big(\mathrm{e}^{S_n} \mathbf{1}_{\{\varepsilon i - b_i < S_i \le \varepsilon i,\ \forall i \le n\}} \Big) \le \mathrm{e}^{\varepsilon n}\, \mathbf{P}\Big\{ \varepsilon i - b_i < S_i \le \varepsilon i,\ \forall i \le n \Big\}. \tag{3.2}$$

To estimate $\varrho_2(\varepsilon)$, we observe that

$$\begin{aligned}\varrho_2(\varepsilon) &\le \sum_{j=1}^n \mathbf{P}\Big\{\exists\,|x|=n:\ H(x)=j,\ V(x_i)\le \varepsilon i,\ \forall i\le n\Big\}\\ &\le \sum_{j=1}^n \mathbf{P}\Big\{\exists\,|x|=n:\ H(x)=j,\ V(x_i)\le \varepsilon i,\ \forall i\le j\Big\}.\end{aligned}$$

Since $\{\exists\,|x|=n:\ H(x)=j,\ V(x_i)\le\varepsilon i,\ \forall i\le j\}\subset\{\exists\,|y|=j:\ H(y)=j,\ V(y_i)\le\varepsilon i,\ \forall i\le j\}$, this yields

$$\varrho_2(\varepsilon)\le \sum_{j=1}^n \mathbf{P}\Big\{\exists\,|y|=j:\ \varepsilon i-b_i<V(y_i)\le\varepsilon i,\ \forall i<j,\ V(y_j)\le \varepsilon j-b_j\Big\}.$$

We can now use the same argument as for $\varrho_1(\varepsilon)$, namely, Chebyshev's inequality and then the change-of-probability formula (2.2), to see that

$$\begin{aligned}\varrho_2(\varepsilon) &\le \sum_{j=1}^n \mathbf{E}\Big(\sum_{|y|=j}\mathbf{1}_{\{\varepsilon i-b_i<V(y_i)\le\varepsilon i,\,\forall i<j,\,V(y_j)\le\varepsilon j-b_j\}}\Big)\\ &= \sum_{j=1}^n \mathbf{E}\Big(\mathrm{e}^{S_j}\mathbf{1}_{\{\varepsilon i-b_i<S_i\le\varepsilon i,\,\forall i<j,\,S_j\le\varepsilon j-b_j\}}\Big)\\ &\le \sum_{j=1}^n \mathrm{e}^{\varepsilon j-b_j}\mathbf{P}\Big\{\varepsilon i-b_i<S_i\le\varepsilon i,\ \forall i<j\Big\}.\end{aligned}$$

Together with (3.2), and recalling that $\varrho(\varepsilon)\le\varrho_1(\varepsilon)+\varrho_2(\varepsilon)$, this yields

$$\begin{aligned}\varrho(\varepsilon) &\le \mathrm{e}^{\varepsilon n}\,\mathbf{P}\Big\{\varepsilon i-b_i<S_i\le\varepsilon i,\ \forall i\le n\Big\}+\sum_{j=1}^n \mathrm{e}^{\varepsilon j-b_j}\mathbf{P}\Big\{\varepsilon i-b_i<S_i\le\varepsilon i,\ \forall i<j\Big\}\\ &= \mathrm{e}^{\varepsilon n}I(n)+\sum_{j=0}^{n-1}\mathrm{e}^{\varepsilon(j+1)-b_{j+1}}I(j),\end{aligned}$$

where $I(0):=1$ and

$$I(j):=\mathbf{P}\Big\{\varepsilon i-b_i<S_i\le\varepsilon i,\ \forall i\le j\Big\},\qquad 1\le j\le n.$$

The idea is now to apply Mogulskii's estimate (2.12) to $I(j)$ for suitably chosen (b_i). Unfortunately, since ε depends on n, we are not allowed to apply (2.12) simultaneously to all $I(j)$, $0\le j\le n$. So let us first work a little bit more, and then apply (2.12) to only a few of the $I(j)$.

We assume that (b_i) is non-increasing. Fix an integer $N \geq 2$, and take $n := kN$ for $k \geq 1$. Then

$$\begin{aligned}
\varrho(\varepsilon) &\leq \mathrm{e}^{\varepsilon kN} I(kN) + \sum_{j=0}^{k-1} \mathrm{e}^{\varepsilon(j+1)-b_{j+1}} I(j) + \sum_{\ell=1}^{N-1} \sum_{j=\ell k}^{(\ell+1)k-1} \mathrm{e}^{\varepsilon(j+1)-b_{j+1}} I(j) \\
&\leq \mathrm{e}^{\varepsilon kN} I(kN) + k \exp(\varepsilon k - b_k) + k \sum_{\ell=1}^{N-1} \exp\Big(\varepsilon(\ell+1)k - b_{(\ell+1)k}\Big) I(\ell k). \qquad (3.3)
\end{aligned}$$

We choose $b_i = b_i(n) := b(n-i)^{1/3} = b(kN-i)^{1/3}$, $0 \leq i \leq n$, and $\varepsilon := \frac{\theta}{n^{2/3}} = \frac{\theta}{(Nk)^{2/3}}$, where $b > 0$ and $\theta > 0$ are constants. By definition, for $1 \leq \ell \leq N$,

$$I(\ell k) = \mathbf{P}\Big\{\theta\Big(\frac{\ell}{N}\Big)^{2/3} \frac{i}{\ell k} - b\Big(\frac{N}{\ell} - \frac{i}{\ell k}\Big)^{1/3} < \frac{S_i}{(\ell k)^{1/3}} \leq \theta\Big(\frac{\ell}{N}\Big)^{2/3} \frac{i}{\ell k},\ \forall i \leq \ell k\Big\}.$$

Applying (2.12) to $g_1(t) := \theta(\frac{\ell}{N})^{2/3} t - b(\frac{N}{\ell} - t)^{1/3}$ and $g_2(t) := \theta(\frac{\ell}{N})^{2/3} t$, we see that, for $1 \leq \ell \leq N$,

$$\limsup_{k\to\infty} \frac{1}{(\ell k)^{1/3}} \log I(\ell k) \leq -\frac{\pi^2\sigma^2}{2b^2} \int_0^1 \frac{\mathrm{d}t}{(\frac{N}{\ell} - t)^{2/3}} = -\frac{3\pi^2\sigma^2}{2b^2} \frac{N^{1/3} - (N-\ell)^{1/3}}{\ell^{1/3}},$$

where σ is the constant in (2.7). Going back to (3.3), we obtain:

$$\limsup_{k\to\infty} \frac{\theta^{1/2}}{(Nk)^{1/3}} \log \varrho\Big(\frac{\theta}{(Nk)^{2/3}}\Big) \leq \theta^{1/2} \alpha_{N,b},$$

where the constant $\alpha_{N,b} = \alpha_{N,b}(\theta)$ is defined by

$$\begin{aligned}
\alpha_{N,b} \ :=\ \max_{1\leq \ell \leq N-1} \Big\{ & \theta - \frac{3\pi^2\sigma^2}{2b^2},\ \frac{\theta}{N} - b(1-\frac{1}{N})^{1/3}, \\
& \frac{\theta(\ell+1)}{N} - b(1-\frac{\ell+1}{N})^{1/3} - \frac{3\pi^2\sigma^2}{2b^2} \frac{N^{1/3} - (N-\ell)^{1/3}}{N^{1/3}} \Big\}.
\end{aligned}$$

Since $\varepsilon \mapsto \varrho(\varepsilon)$ is non-increasing, this yields

$$\limsup_{\varepsilon\to 0} \varepsilon^{1/2} \log \varrho(\varepsilon) \leq \theta^{1/2} \alpha_{N,b}.$$

We let $N \to \infty$. By definition,

$$\limsup_{N\to\infty} \alpha_{N,b} \leq \max\Big\{\theta - \frac{3\pi^2\sigma^2}{2b^2},\ -b,\ f(\theta, b)\Big\},$$

where $f(\theta, b) := \sup_{t\in(0,\,1]} \{\theta t - b(1-t)^{1/3} - \frac{3\pi^2\sigma^2}{2b^2}[1-(1-t)^{1/3}]\}$.

Elementary computations show that as long as $b < \frac{3\pi^2\sigma^2}{2b^2} \le b + 3\theta$, we have $f(\theta, b) = \theta - \frac{3\pi^2\sigma^2}{2b^2} + \frac{2}{3(3\theta)^{1/2}}(\frac{3\pi^2\sigma^2}{2b^2} - b)^{3/2}$. Thus $\max\{\theta - \frac{3\pi^2\sigma^2}{2b^2},\ -b,\ f(\theta, b)\} = \max\{f(\theta, b),\ -b\}$, which equals $-b$ if $\theta = \frac{\pi^2\sigma^2}{2b^2} - \frac{b}{3}$. As a consequence, for any $b > 0$ satisfying $b < \frac{3\pi^2\sigma^2}{2b^2}$,

$$\limsup_{\varepsilon \to 0} \varepsilon^{1/2} \log \varrho(\varepsilon) \le -b\sqrt{\frac{\pi^2\sigma^2}{2b^2} - \frac{b}{3}} = -\sqrt{\frac{\pi^2\sigma^2}{2} - \frac{b^3}{3}}.$$

Letting $b \to 0$, this yields (3.1) and completes the proof of the upper bound in Theorem 1.2. $\square$

4 Proof of Theorem 1.2: the lower bound

Before proceeding to the proof of the lower bound in Theorem 1.2, we recall two inequalities: the first gives a useful lower tail estimate for the number of individuals in a super-critical Galton–Watson process conditioned on survival, whereas the second concerns an elementary property of the conditional distribution of a sum of independent random variables. Let us recall that Z_n is the number of particles in the n-th generation.

Fact 4.1 (McDiarmid [17]) *There exists $\vartheta > 1$ such that*

$$\mathbf{P}\{Z_n \le \vartheta^n \,|\, Z_n > 0\} \le \vartheta^{-n}, \qquad \forall n \ge 1. \tag{4.1}$$

Fact 4.2 ([9]) *If $X_1, X_2, \ldots, X_N$ are independent non-negative random variables, and if $F : (0, \infty) \to \mathbb{R}_+$ is non-increasing, then*

$$\mathbf{E}\Big[F\Big(\sum_{i=1}^N X_i\Big) \,\Big|\, \sum_{i=1}^N X_i > 0\Big] \le \max_{1\le i\le N} \mathbf{E}[F(X_i) \,|\, X_i > 0].$$

This section is devoted to the proof of the lower bound in (2.6):

$$\liminf_{\varepsilon \to 0} \varepsilon^{1/2} \log \varrho(\varepsilon) \ge -\frac{\pi\sigma}{2^{1/2}}, \tag{4.2}$$

where $\varrho(\varepsilon)$ and σ are as in (2.5) and (2.7), respectively.

The basic idea consists in constructing a new Galton–Watson tree $\mathbb{G} = \mathbb{G}(\varepsilon)$ within the branching random walk, and obtaining a lower bound for $\varrho(\varepsilon)$ in terms of $\mathbb{G}$.

Recall from (1.7) that conditioned on survival, $\frac{1}{j}\max_{|z|\le j} V(z)$ converges almost surely, for $j \to \infty$, to a finite constant. [The fact that this limiting constant is finite is a consequence of

$\mathbf{E}(\sum_{|x|=1} \mathrm{e}^{\delta_2 V(x)}) < \infty$ in (2.4).] Since the system survives with (strictly) positive probability, we can fix a sufficiently large constant $M > 0$ such that

$$\text{(4.3)} \quad \inf_{j\ge 0} \mathbf{P}\Big\{ \max_{|x|\le j} V(x) \le Mj \Big\} \ge \frac{1}{2}, \qquad \kappa := \inf_{j\ge 0} \mathbf{P}\Big\{ Z_j > 0,\ \max_{|x|\le j} V(x) \le Mj \Big\} > 0,$$

where, as before, $Z_j := \#\{x : |x| = j\}$.

Fix a constant $0 < \alpha < 1$. For any integers $n > L \ge 1$ with $(1-\alpha)\varepsilon L \ge M(n-L)$, we consider the set $G_{n,\varepsilon} = G_{n,\varepsilon}(L)$ defined by [4]

$$G_{n,\varepsilon} := \{|x| = n :\ V(x_i) \le \alpha\varepsilon i, \text{ for } 1 \le i \le L;\ \max_{z > x_L :\, |z| \le n} [V(z) - V(x_L)] \le (1-\alpha)\varepsilon L\}.$$

By definition, for any $x \in G_{n,\varepsilon}$, we have $V(x_i) \le \varepsilon i$, for $1 \le i \le n$.

If $G_{n,\varepsilon} \ne \emptyset$, the elements of $G_{n,\varepsilon}$ form the first generation of the new Galton–Watson tree $\mathbb{G}_{n,\varepsilon}$, and we construct $\mathbb{G}_{n,\varepsilon}$ by iterating the same procedure: for example, the second generation in $\mathbb{G}_{n,\varepsilon}$ consists of y with $|y| = 2n$ being a descendant of some $x \in G_{n,\varepsilon}$ such that $V(y_{n+i}) - V(x) \le \alpha\varepsilon i$, for $1 \le i \le L$ and $\max_{z > y_{n+L} :\, |z| \le 2n} [V(z) - V(y_{n+L})] \le (1-\alpha)\varepsilon L$.

Let $q_{n,\varepsilon}$ denote the probability of extinction of the Galton–Watson tree $\mathbb{G}_{n,\varepsilon}$. It is clear that

$$\varrho(\varepsilon) \ge 1 - q_{n,\varepsilon},$$

so we only need to find a lower bound for $1 - q_{n,\varepsilon}$. In order to do so, we introduce, for $b \in \mathbb{R}$ and $n \ge 1$,

$$\text{(4.4)} \qquad \varrho(b,\, n) := \mathbf{P}\Big\{ \exists |x| = n :\ V(x_i) \le bi, \text{ for } 1 \le i \le n \Big\}.$$

Let us first prove some preliminary results.

Lemma 4.3 *Let $0 < \alpha < 1$ and $\varepsilon > 0$. Let $n > L \ge 1$ be such that $(1-\alpha)\varepsilon L \ge M(n-L)$. Then*

$$\mathbf{P}\{G_{n,\varepsilon} \ne \emptyset\} \ge \frac{1}{2} \varrho(\alpha\varepsilon,\, n).$$

Proof. By definition,

$$\mathbf{P}\{G_{n,\varepsilon} \ne \emptyset\} = \mathbf{E}\Big(\mathbf{1}_{\{\exists |y| = L :\, V(y_i) \le \alpha\varepsilon i,\, \forall i \le L\}} \mathbf{P}\Big\{ \max_{|z| \le n-L} V(z) \le (1-\alpha)\varepsilon L \Big\} \Big).$$

Since $(1-\alpha)\varepsilon L \ge M(n-L)$, it follows from (4.3) that

$$\mathbf{P}\{G_{n,\varepsilon} \ne \emptyset\} \ge \frac{1}{2} \mathbf{P}\{\exists |y| = L :\ V(y_i) \le \alpha\varepsilon i,\ \forall i \le L\},$$

and the r.h.s. is at least $\frac{1}{2}\varrho(\alpha\varepsilon,\, n)$. $\square$

[4] We write $z > x$ if x is an ancestor of z.

Lemma 4.4 *Let $0 < \alpha < 1$ and $\varepsilon > 0$. Let $n > L \ge 1$ be such that $(1-\alpha)\varepsilon L \ge M(n-L)$. We have*

$$\mathbf{P}\{1 \le \# G_{n,\varepsilon} \le \vartheta^{n-L}\} \le \frac{1}{\kappa\, \vartheta^{n-L}}, \tag{4.5}$$

where $\kappa > 0$ and $\vartheta > 1$ are the constants in (4.3) and (4.1), respectively.

Proof. By definition,

$$\# G_{n,\varepsilon} = \sum_{|x|=L} \eta_x \mathbf{1}_{\{V(x_i) \le \alpha\varepsilon i,\, \forall i \le L\}},$$

where

$$\eta_x := \#\{y > x:\ |y| = n\}\, \mathbf{1}_{\{\max_{\{z>x:\, |z|\le n\}}[V(z)-V(x)] \le (1-\alpha)\varepsilon L\}}.$$

By Fact 4.2, for any $\ell \ge 1$, with $F(x) = \mathbf{1}_{\{x \le \ell\}}$,

$$\mathbf{P}\Big\{\# G_{n,\varepsilon} \le \ell \,\Big|\, \# G_{n,\varepsilon} > 0\Big\} \le \mathbf{P}\Big\{Z_{n-L} \le \ell \,\Big|\, Z_{n-L} > 0,\ \max_{|z| \le n-L} V(z) \le (1-\alpha)\varepsilon L\Big\},$$

where, as before, $Z_{n-L} := \#\{|x| = n-L\}$. Since $(1-\alpha)\varepsilon L \ge M(n-L)$, it follows from (4.3) that $\mathbf{P}\{Z_{n-L} > 0,\ \max_{|z|\le n-L} V(z) \le (1-\alpha)\varepsilon L\} \ge \kappa > 0$. Therefore,

$$\mathbf{P}\{1 \le \# G_{n,\varepsilon} \le \ell\} \le \frac{1}{\kappa}\mathbf{P}\Big\{Z_{n-L} \le \ell \,\Big|\, Z_{n-L} > 0\Big\}.$$

This implies (4.5) by means of Fact 4.1. $\square$

To state the next estimate, we recall that $\nu(x)$ is the number of children of x, and that $(S_i - S_{i-1},\, \nu_{i-1})$, $i \ge 1$, are i.i.d. random vectors (with $S_0 := 0$) whose common distribution is given by (2.10).

Lemma 4.5 *Let $n \ge 1$. For any $1 \le i \le n$, let $I_{i,n} \subset \mathbb{R}$ be a Borel set. Let $r_n \ge 1$ be an integer. We have*

$$\mathbf{P}\Big\{\exists |x| = n:\ V(x_i) \in I_{i,n}\,,\ \forall 1 \le i \le n\Big\} \ge \frac{\mathbf{E}[\mathrm{e}^{S_n}\, \mathbf{1}_{\{S_i \in I_{i,n}\,,\, \nu_{i-1} \le r_n,\, \forall\, 1 \le i \le n\}}]}{1 + (r_n - 1)\sum_{j=1}^n h_{j,n}},$$

where

$$h_{j,n} := \sup_{u \in I_{j,n}} \mathbf{E}\Big(\mathrm{e}^{S_{n-j}} \mathbf{1}_{\{S_\ell \in I_{\ell+j,n} - u,\, \forall 0 \le \ell \le n-j\}}\Big), \tag{4.6}$$

and $I_{\ell+j,n} - u := \{v - u:\ v \in I_{\ell+j,n}\}$.

Proof. Let

$$Y_n := \sum_{|x|=n} \mathbf{1}_{\{V(x_i)\in I_{i,n}\,,\, \nu(x_{i-1})\le r_n,\, \forall 1\le i\le n\}}.$$

By definition,

$$\begin{aligned}
\mathbf{E}(Y_n^2) &= \mathbf{E}\Big(\sum_{|x|=n}\sum_{|y|=n} \mathbf{1}_{\{V(x_i)\in I_{i,n}\,,\, \nu(x_{i-1})\le r_n,\, V(y_i)\in I_{i,n}\,,\, \nu(y_{i-1})\le r_n,\, \forall 1\le i\le n\}}\Big) \\
&= \mathbf{E}(Y_n) + \mathbf{E}\Big(\sum_{j=0}^{n-1}\sum_{|z|=j} \mathbf{1}_{\{V(z_i)\in I_{i,n}\,,\, \nu(z_{i-1})\le r_n,\, \forall i\le j\}} D_{j+1,n}(z)\Big), \qquad (4.7)
\end{aligned}$$

with

$$\begin{aligned}
D_{j+1,n}(z) &:= \sum_{(x_{j+1},\, y_{j+1})}\sum_{(x,\, y)} \mathbf{1}_{\{V(x_i)\in I_{i,n}\,,\, \nu(x_{i-1})\le r_n,\, V(y_i)\in I_{i,n}\,,\, \nu(y_{i-1})\le r_n,\, \forall j+1\le i\le n\}} \\
&\le \sum_{(x_{j+1},\, y_{j+1})}\sum_{(x,\, y)} \mathbf{1}_{\{V(x_i)\in I_{i,n}\,,\, \nu(x_{i-1})\le r_n,\, V(y_i)\in I_{i,n}\,,\, \forall j+1\le i\le n\}},
\end{aligned}$$

where the double sum $\sum_{(x_{j+1},\, y_{j+1})}$ is over pairs $(x_{j+1},\, y_{j+1})$ of distinct children of z (thus $|x_{j+1}| = |y_{j+1}| = j+1$), while $\sum_{(x,\, y)}$ is over pairs $(x,\, y)$ with $|x| = |y| = n$ such that[5] $x \ge x_{j+1}$ and $y \ge y_{j+1}$.

The $\mathbf{E}[\sum_{j=0}^{n-1}\sum_{|z|=j} \mathbf{1}_{\{\cdots\}} D_{j+1,n}(z)]$ expression on the right-hand side of (4.7) is bounded by

$$\mathbf{E}\Big(\sum_{j=0}^{n-1}\sum_{|z|=j} \mathbf{1}_{\{V(z_i)\in I_{i,n}\,,\, \nu(z_{i-1})\le r_n,\, \forall i\le j\}} \sum_{(x_{j+1},\, y_{j+1})}\sum_{x} \mathbf{1}_{\{V(x_i)\in I_{i,n}\,,\, \nu(x_{i-1})\le r_n,\, \forall j+1\le i\le n\}}\, h_{j+1,n}\Big),$$

where $h_{j+1,n} := \sup_{u\in I_{j+1,n}} \mathbf{E}[\sum_{|y|=n-j-1} \mathbf{1}_{\{V(y_\ell)\in I_{\ell+j+1,n}-u,\, \forall 0\le \ell\le n-j-1\}}]$, which is in agreement with (4.6), thanks to the change of probability formula (2.8). [The sum $\sum_x$ is, of course, still over x with $|x| = n$ such that $x \ge x_{j+1}$.]

Thanks to the condition $\nu(x_j) \le r_n$ (i.e., $\nu(z) \le r_n$), we see that the sum $\sum_{y_{j+1}}$ in the last display gives at most a factor of $r_n - 1$; which yields that the last display is at most $(r_n-1)\mathbf{E}(\sum_{j=0}^{n-1} Y_n h_{j+1,n})$. In other words, we have proved that

$$\mathbf{E}\Big(\sum_{j=0}^{n-1}\sum_{|z|=j} \mathbf{1}_{\{V(z_i)\in I_{i,n}\,,\, \nu(z_{i-1})\le r_n,\, \forall i\le j\}} D_{j+1,n}(z)\Big) \le (r_n - 1)\sum_{j=0}^{n-1} \mathbf{E}(Y_n) h_{j+1,n}.$$

[5] We write $y \ge x$ if either $y > x$ or $y = x$.

This yields $\mathbf{E}(Y_n^2) \le [1 + (r_n - 1)\sum_{j=0}^{n-1} h_{j+1,n}]\mathbf{E}(Y_n)$. Therefore,

$$\text{(4.8)} \qquad \frac{\mathbf{E}(Y_n^2)}{[\mathbf{E}(Y_n)]^2} \le \frac{1 + (r_n - 1)\sum_{j=1}^{n} h_{j,n}}{\mathbf{E}(Y_n)} = \frac{1 + (r_n - 1)\sum_{j=1}^{n} h_{j,n}}{\mathbf{E}(\mathrm{e}^{S_n}\,\mathbf{1}_{\{S_i \in I_{i,n}\,,\ \nu_{i-1} \le r_n,\ \forall 1 \le i \le n\}})},$$

the last inequality being a consequence of (2.9). By the Cauchy–Schwarz inequality, $\mathbf{P}\{Y_n \ge 1\} \ge \frac{[\mathbf{E}(Y_n)]^2}{\mathbf{E}(Y_n^2)}$. Recalling the definition of Y_n, we obtain from (4.8) that

$$\begin{aligned} & \mathbf{P}\Big\{\exists |x| = n:\ V(x_i) \in I_{i,n}\,,\ \nu(x_{i-1}) \le r_n,\ \forall 1 \le i \le n\Big\} \\ \text{(4.9)} \qquad \ge\ & \frac{\mathbf{E}[\mathrm{e}^{S_n}\,\mathbf{1}_{\{S_i \in I_{i,n}\,,\ \nu_{i-1} \le r_n,\ \forall 1 \le i \le n\}}]}{1 + (r_n - 1)\sum_{j=1}^{n} h_{j,n}}. \end{aligned}$$

Lemma 4.5 follows immediately from (4.9). □

The key step in the proof of the lower bound in Theorem 1.2 is the following estimate.

Lemma 4.6 *For any $\theta > 0$,*

$$\liminf_{n\to\infty} \frac{\log \varrho(\theta n^{-2/3},\, n)}{n^{1/3}} \ge -\frac{\pi\sigma}{(2\theta)^{1/2}},$$

where $\sigma > 0$ is the constant in (2.7).

Proof. Let $0 < \lambda < \frac{\pi\sigma}{(2\theta)^{1/2}}$, and let $I_{i,n} := [\frac{\theta i}{n^{2/3}} - \lambda\, n^{1/3},\ \frac{\theta i}{n^{2/3}}]$ (for $1 \le i \le n$). Since $\varrho(\theta n^{-2/3},\, n) \ge \mathbf{P}\{\exists |x| = n:\ V(x_i) \in I_{i,n}\,,\ \forall 1 \le i \le n\}$, it follows from Lemma 4.5 that for any integer $r_n \ge 1$,

$$\varrho(\theta n^{-2/3},\, n) \ge \frac{\mathbf{E}[\mathrm{e}^{S_n}\,\mathbf{1}_{\{S_i \in I_{i,n}\,,\ \nu_{i-1} \le r_n,\ \forall 1 \le i \le n\}}]}{1 + (r_n - 1)\sum_{j=1}^{n} h_{j,n}} =: \frac{\Lambda_n}{1 + (r_n - 1)\sum_{j=1}^{n} h_{j,n}},$$

where $h_{j,n}$ is defined in (4.6), while $(S_i - S_{i-1},\, \nu_{i-1})$, $i \ge 1$, are i.i.d. random vectors (with $S_0 := 0$) whose common distribution is given by (2.10).

For any $\theta_1 < \theta$, we have

$$\begin{aligned} \Lambda_n &\ge \mathrm{e}^{\theta_1 n^{1/3}}\,\mathbf{P}\{S_i \in I_{i,n}\,,\ \nu_{i-1} \le r_n,\ \forall 1 \le i \le n,\ S_n \ge \theta_1 n^{1/3}\} \\ &= \mathrm{e}^{\theta_1 n^{1/3}}\,\mathbf{P}\Big\{\theta\frac{i}{n} - \lambda \le \frac{S_i}{n^{1/3}} \le \theta\frac{i}{n},\ \nu_{i-1} \le r_n,\ \forall 1 \le i \le n,\ \frac{S_n}{n^{1/3}} \ge \theta_1\Big\}. \end{aligned}$$

For any $n \ge 1$, we consider i.i.d. random variables $X_i^{(n)}$, $1 \le i \le n$, having the same distribution as S_1 conditioned on $\nu_0 \le r_n$. Let $S_0^{(n)} = 0$ and $S_i := X_1^{(n)} + \cdots + X_i^{(n)}$ for $1 \le i \le n$. Then

$$\Lambda_n \ge \mathrm{e}^{\theta_1 n^{1/3}}\,[\mathbf{P}\{\nu_0 \le r_n\}]^n\,\mathbf{P}\Big\{\theta\frac{i}{n} - \lambda \le \frac{S_i^{(n)}}{n^{1/3}} \le \theta\frac{i}{n},\ \forall 1 \le i \le n,\ \frac{S_n^{(n)}}{n^{1/3}} \ge \theta_1\Big\}.$$

We now choose $r_n := \lfloor e^{n^{1/4}} \rfloor$. By definition, $\mathbf{P}\{\nu_0 > r_n\} = \mathbf{E}(\sum_{|x|=1} e^{-V(x)} \mathbf{1}_{\{Z_1 > r_n\}})$, where $Z_1 = \sum_{|y|=1} 1$ as before. By Markov's inequality, $\mathbf{P}\{Z_1 > r_n\} \le \frac{\mathbf{E}(Z_1^{1+\delta})}{r_n^{1+\delta}}$. Since $\mathbf{E}(Z_1^{1+\delta}) < \infty$ (Condition (1.3)) and $\mathbf{E}(\sum_{|x|=1} e^{-(1+\delta_1)V(x)}) < \infty$ (see (2.3)), an application of Hölder's inequality confirms that $\mathbf{P}\{\nu_0 > r_n\} \le r_n^{-\delta_4}$ for some $\delta_4 > 0$ and all sufficiently large n. In view of our choice of r_n, we see that $[\mathbf{P}\{\nu_0 \le r_n\}]^n \to 1$. Therefore, for all sufficiently large n,

$$\Lambda_n \ge \frac{1}{2} e^{\theta_1 n^{1/3}} \mathbf{P}\Big\{\theta \frac{i}{n} - \lambda \le \frac{S_i^{(n)}}{n^{1/3}} \le \theta \frac{i}{n}, \ \forall 1 \le i \le n, \ \frac{S_n^{(n)}}{n^{1/3}} \ge \theta_1 \Big\}.$$

To deal with the probability expression on the right-hand side, we intend to apply (2.13); so we need to check condition (2.11). Recall that S_1 has finite exponential moments in the neighbourhood of 0. Thus, the first condition in (2.11), namely, $\sup_{n\ge 1} \mathbf{E}(|X_1^{(n)}|^{2+\eta}) < \infty$ for some $\eta > 0$, is trivially satisfied. To check the second condition, we see that since $\mathbf{E}(S_1) = 0$, we have $\mathbf{E}(X_1^{(n)}) = -\frac{\mathbf{E}[S_1 \mathbf{1}_{\{\nu_0 > r_n\}}]}{\mathbf{P}\{\nu_0 \le r_n\}}$. Since $\mathbf{P}\{\nu_0 > r_n\} \le r_n^{-\delta_4}$ for some $\delta_4 > 0$ and all sufficiently large n, and since S_1 has some finite exponential moments, the second condition in (2.11), $\mathbf{E}(X_1^{(n)}) = o(\frac{a_n}{n})$, is also satisfied (regardless of the value of the sequence $a_n \to \infty$) in view of the Cauchy–Schwarz inequality. Moreover, $\mathbf{E}(X_1^{(n)}) \to 0$, which yields $\mathrm{Var}(X_1^{(n)}) \to \mathbf{E}(S_1^2) - 0 = \sigma^2$: the third and last condition in (2.11) is verified.

We are therefore entitled to apply (2.13): taking $g_1(t) := \theta t - \lambda$ and $g_2(t) := \theta t$, we see that for any $\lambda_1 \in (0, \lambda)$ and all sufficiently large n,

$$\Lambda_n \ge \frac{1}{2} e^{\theta_1 n^{1/3}} \exp\Big(- \frac{\pi^2 \sigma^2}{2\lambda_1^2} n^{1/3} \Big),$$

which implies, for all sufficiently large n,

$$\varrho(\theta n^{-2/3}, n) \ge \frac{\frac{1}{2} \exp[(\theta_1 - \frac{\pi^2\sigma^2}{2\lambda_1^2}) n^{1/3}]}{1 + (r_n - 1)\sum_{j=1}^n h_{j,n}}. \tag{4.10}$$

To estimate $\sum_{j=1}^n h_{j,n}$, we observe that

$$\begin{aligned} h_{j,n} &= \sup_{u \in I_{j,n}} \mathbf{E}\Big(e^{S_{n-j}} \mathbf{1}_{\{S_i \in [\frac{\theta(i+j)}{n^{2/3}} - \lambda n^{1/3} - u, \, \frac{\theta(i+j)}{n^{2/3}} - u], \, \forall 0 \le i \le n-j\}} \Big) \\ &= \sup_{v \in [0, \, \lambda n^{1/3}]} \mathbf{E}\Big(e^{S_{n-j}} \mathbf{1}_{\{S_i \in [\frac{\theta i}{n^{2/3}} - \lambda n^{1/3} + v, \, \frac{\theta i}{n^{2/3}} + v], \, \forall 0 \le i \le n-j\}} \Big) \\ &\le e^{\theta(n-j)n^{-2/3} + \lambda n^{1/3}} \sup_{v \in [0, \, \lambda n^{1/3}]} \mathbf{P}\Big\{ \frac{\theta i}{n^{2/3}} - \lambda n^{1/3} + v \le S_i \le \frac{\theta i}{n^{2/3}} + v, \ \forall 0 \le i \le n-j \Big\}. \end{aligned}$$

We now use the same trick as in the proof of the upper bound in Theorem 1.2 by sending n to infinity along a subsequence. Fix an integer $N \geq 1$. Let $n := Nk$, with $k \geq 1$. For any $j \in [(\ell-1)k+1,\, \ell k] \cap \mathbb{Z}$ (with $1 \leq \ell \leq N$), we have

$$h_{j,n} \leq \mathrm{e}^{\theta(N-\ell+1)kn^{-2/3}+\lambda n^{1/3}} \sup_{v\in[0,\, \lambda n^{1/3}]} \mathbf{P}\Big\{ v - \lambda n^{1/3} \leq S_i - \frac{\theta i}{n^{2/3}} \leq v,\ \forall i \leq (N-\ell)k \Big\}.$$

Unfortunately, the interval $[0,\, \lambda n^{1/3}]$ in $\sup_{v\in[0,\, \lambda n^{1/3}]} \mathbf{P}\{\cdots\}$ is very large, so we split it into smaller ones of type $[\frac{(m-1)\lambda n^{1/3}}{N},\ \frac{m\lambda n^{1/3}}{N}]$ (for $1 \leq m \leq N$), to see that the $\sup_{v\in[0,\, \lambda n^{1/3}]} \mathbf{P}\{\cdots\}$ expression is

$$\begin{aligned} &\leq \max_{1\leq m\leq N} \mathbf{P}\Big\{ \frac{(m-1)\lambda n^{1/3}}{N} - \lambda n^{1/3} \leq S_i - \frac{\theta i}{n^{2/3}} \leq \frac{m\lambda n^{1/3}}{N},\ \forall i \leq (N-\ell)k \Big\} \\ &= \max_{1\leq m\leq N} \mathbf{P}\Big\{ -\frac{(N-m+1)\lambda}{N^{2/3}} \leq \frac{S_i}{k^{1/3}} - \frac{\theta}{N^{2/3}}\frac{i}{k} \leq \frac{m\lambda}{N^{2/3}},\ \forall i \leq (N-\ell)k \Big\}. \end{aligned}$$

We are now entitled to apply (2.12) to $n := (N-\ell)k$, $g_1(t) := \frac{\theta}{(N-\ell)^{1/3}N^{2/3}}t - \frac{(N-m+1)\lambda}{(N-\ell)^{1/3}N^{2/3}}$ and $g_2(t) := \frac{\theta}{(N-\ell)^{1/3}N^{2/3}}t + \frac{m\lambda}{(N-\ell)^{1/3}N^{2/3}}$, to see that for any $1 \leq \ell \leq N$ and uniformly in $j \in [(\ell-1)k+1,\, \ell k] \cap \mathbb{Z}$ (and in $j=0$, which formally corresponds to $\ell = 0$),

$$\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log h_{j,Nk} \leq \frac{\theta(N-\ell+1)}{N} + \lambda - \frac{\pi^2\sigma^2}{2}\frac{(N-\ell)N}{(N+1)^2\lambda^2},$$

which is bounded by $\frac{\theta(N+1)}{N} + \lambda - \frac{\pi^2\sigma^2}{2}\frac{N^2}{(N+1)^2\lambda^2}$ (recalling that $\theta > \frac{\pi^2\sigma^2}{2\lambda^2}$). As a consequence,

$$\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log \sum_{j=0}^{n} h_{j,Nk} \leq \frac{\theta(N+1)}{N} + \lambda - \frac{\pi^2\sigma^2}{2}\frac{N^2}{(N+1)^2\lambda^2} =: c(\theta, N, \lambda).$$

Going back to (4.10), we get

$$\liminf_{k\to\infty} \frac{\log \varrho(\theta N^{-2/3}k^{-2/3},\, Nk)}{N^{1/3}k^{1/3}} \geq \theta_1 - \frac{\pi^2\sigma^2}{2\lambda_1^2} - c(\theta, N, \lambda).$$

By the monotonicity of $n \mapsto \varrho(\theta n^{-2/3},\, n)$, we obtain:

$$\liminf_{n\to\infty} \frac{\log \varrho(\theta n^{-2/3},\, n)}{n^{1/3}} \geq \theta_1 - \frac{\pi^2\sigma^2}{2\lambda_1^2} - c(\theta, N, \lambda).$$

Sending $N \to \infty$, $\theta_1 \to \theta$, $\lambda \to \frac{\pi\sigma}{(2\theta)^{1/2}}$ and $\lambda_1 \to \frac{\pi\sigma}{(2\theta)^{1/2}}$ (in this order) completes the proof of Lemma 4.6. $\square$

We now have all the ingredients for the proof of the lower bound in Theorem 1.2.

Proof of Theorem 1.2: the lower bound. Fix constants $0 < \alpha < 1$ and $b > \max\{\frac{M}{1-\alpha}, \frac{(3\pi\sigma)^2}{\alpha(\log\vartheta)^2}\}$. Let $n > 1$. Let

$$\varepsilon = \varepsilon(n) := \frac{b}{n^{2/3}}, \qquad L = L(n) := n - \lfloor n^{1/3} \rfloor.$$

Then $(1-\alpha)\varepsilon L \geq M(n-L)$ for all sufficiently large n, say[6] $n \geq n_0$.

Consider the moment generating function of the reproduction distribution in the Galton–Watson tree $\mathbb{G}_{n,\varepsilon}$:

$$f(s) := \mathbf{E}(s^{\# G_{n,\varepsilon}}), \qquad s \in [0,\, 1].$$

It is well-known that $q_{n,\varepsilon}$, the extinction probability of $\mathbb{G}_{n,\varepsilon}$, satisfies $q_{n,\varepsilon} = f(q_{n,\varepsilon})$. Therefore, for any $0 < r < \min\{q_{n,\varepsilon}\,,\, \frac{1}{16}\}$,

$$q_{n,\varepsilon} = f(0) + \int_0^{q_{n,\varepsilon}} f'(s)\,\mathrm{d}s = f(0) + \int_0^{q_{n,\varepsilon}-r} f'(s)\,\mathrm{d}s + \int_{q_{n,\varepsilon}-r}^{q_{n,\varepsilon}} f'(s)\,\mathrm{d}s.$$

Since $s \mapsto f'(s)$ is non-decreasing on $[0,\, 1]$, we have $\int_0^{q_{n,\varepsilon}-r} f'(s)\,\mathrm{d}s \leq f'(1-r)$. On the other hand, since $f'(s) \leq f'(q_{n,\varepsilon}) \leq 1$ for $s \in [0,\, q_{n,\varepsilon}]$, we have $\int_{q_{n,\varepsilon}-r}^{q_{n,\varepsilon}} f'(s)\,\mathrm{d}s \leq r$. Therefore,

$$q_{n,\varepsilon} \leq f(0) + f'(1-r) + r.$$

Of course, $f(0) = \mathbf{P}\{G_{n,\varepsilon} = \emptyset\}$, whereas $f'(1-r) = \mathbf{E}[(\# G_{n,\varepsilon})(1-r)^{\# G_{n,\varepsilon}-1}]$, which is bounded by $\frac{1}{1-r}\mathbf{E}[(\# G_{n,\varepsilon})\mathrm{e}^{-r\# G_{n,\varepsilon}}]$ (using the elementary inequality $1-u \leq \mathrm{e}^{-u}$ for $u \geq 0$). This leads to (recalling that $r < \frac{1}{16} < \frac{1}{2}$):

$$1 - q_{n,\varepsilon} \geq \mathbf{P}\{G_{n,\varepsilon} \neq \emptyset\} - 2\mathbf{E}[(\# G_{n,\varepsilon})\mathrm{e}^{-r\# G_{n,\varepsilon}}] - r.$$

Since $u \mapsto u\mathrm{e}^{-ru}$ is decreasing on $[\frac{1}{r},\, \infty)$, we see that $\mathbf{E}[(\# G_{n,\varepsilon})\mathrm{e}^{-r\# G_{n,\varepsilon}}]$ is bounded by $\mathbf{E}[(\# G_{n,\varepsilon})\mathrm{e}^{-r\# G_{n,\varepsilon}}\,\mathbf{1}_{\{\# G_{n,\varepsilon} \leq r^{-2}\}}] + r^{-2}\mathrm{e}^{-1/r} \leq r^{-2}\mathbf{P}\{1 \leq \# G_{n,\varepsilon} \leq r^{-2}\} + r^{-2}\mathrm{e}^{-1/r}$. Accordingly,

$$\begin{aligned} 1 - q_{n,\varepsilon} &\geq \mathbf{P}\{G_{n,\varepsilon} \neq \emptyset\} - \frac{2}{r^2}\mathbf{P}\{1 \leq \# G_{n,\varepsilon} \leq r^{-2}\} - \frac{2\mathrm{e}^{-1/r}}{r^2} - r \\ &\geq \frac{1}{2}\,\varrho(\alpha\varepsilon,\, n) - \frac{2}{r^2}\mathbf{P}\{1 \leq \# G_{n,\varepsilon} \leq r^{-2}\} - 2r, \end{aligned}$$

the last inequality following from Lemma 4.3 and the fact that $\sup_{\{0<r\leq\frac{1}{16}\}} \frac{1}{r^3}\mathrm{e}^{-1/r} < \frac{1}{2}$.

We choose $r := \frac{1}{16}\,\varrho(\alpha\varepsilon,\, n)$. [Since $\varrho(\varepsilon) \geq 1 - q_{n,\varepsilon}$, whereas $\lim_{\varepsilon\to 0}\varrho(\varepsilon) = 0$ (proved in Section 3), we have $q_{n,\varepsilon} \to 1$ for $n \to \infty$, and thus the requirement $0 < r < \min\{q_{n,\varepsilon}\,,\, \frac{1}{16}\}$ is satisfied for all sufficiently large n.]

[6]Without further mention, the value of n_0 can change from line to line when other conditions are to be satisfied.

By Lemma 4.6, $r^{-2} \le \vartheta^{n-L}$ for all $n \ge n_0$ (because $\frac{2\pi\sigma}{(\alpha b)^{1/2}} < \log\vartheta$ by our choice of b). Therefore, an application of Lemma 4.4 tells us that for $n \ge n_0$, $\mathbf{P}\{1 \le \#G_{n,\varepsilon} \le r^{-2}\} \le \frac{1}{\kappa\,\vartheta^{n-L}}$, which, by Lemma 4.4 again, is bounded by r^3 (because $\frac{3\pi\sigma}{(\alpha b)^{1/2}} < \log\vartheta$). Consequently, for all $n \ge n_0$,

$$1 - q_{n,\varepsilon} \ge \frac{1}{2}\,\varrho(\alpha\varepsilon,\, n) - 2r - 2r = \frac{1}{4}\,\varrho(\alpha\varepsilon,\, n).$$

Recall that $\varrho(\varepsilon) \ge 1 - q_{n,\varepsilon}$. Therefore,

$$\liminf_{n\to\infty} \frac{1}{n^{1/3}} \log \varrho\Big(\frac{b}{n^{2/3}}\Big) \ge -\frac{\pi\sigma}{(2\alpha b)^{1/2}}.$$

Since $\varepsilon \mapsto \varrho(\varepsilon)$ is non-increasing, we obtain:

$$\liminf_{\varepsilon\to 0} \varepsilon^{1/2} \log \varrho(\varepsilon) \ge -\frac{\pi\sigma}{(2\alpha)^{1/2}}.$$

Sending $\alpha \to 1$ yields (4.2), and thus proves the lower bound in Theorem 1.2. $\square$

5 Appendix. Proof of Lemma 2.1

We write $S_j^{(n)} := \sum_{i=1}^{j} X_i^{(n)}$ (for $1 \le j \le n$) and $S_0^{(n)} := 0$. We need to prove the lower bound in (2.13), and the upper bound in (2.12).

Lower bound in (2.13). We want to prove that for any $b > 0$,

$$\liminf_{n\to\infty} \frac{a_n^2}{n} \log \mathbf{P}\Big\{E_n,\ \frac{S_n^{(n)}}{a_n} \ge g_2(1) - b\Big\} \ge -\frac{\pi^2\sigma^2}{2}\int_0^1 \frac{\mathrm{d}t}{[g_2(t) - g_1(t)]^2}.$$

Let $g:\ [0,\,1] \to \mathbb{R}$ be a continuous function such that $g_1(t) < g(t) < g_2(t)$ for all $t \in [0,\,1]$. It suffices to prove the lower bound in (2.13) when $b > 0$ is sufficiently small; so we assume, without loss of generality, that $g(1) \ge g_2(1) - b$.

Let $\delta > 0$ be such that

$$g(t) - g_1(t) > 3\delta, \quad g_2(t) - g(t) > 9\delta, \qquad \forall t \in [0,\,1]. \tag{5.1}$$

Let A be a sufficiently large integer such that

$$\sup_{0\le s\le t\le 1:\ t-s\le \frac{2}{A}} \big(|g_1(t) - g_1(s)| + |g(t) - g(s)| + |g_2(t) - g_2(s)|\big) \le \delta. \tag{5.2}$$

Let $r_n := \lfloor A a_n^2 \rfloor$, $N = N(n) := \lfloor \frac{n}{r_n} \rfloor$. Let $m_N := n$ and $m_k := k r_n$ for $0 \le k \le N-1$.

Since $g(1) \geq g_2(1) - b$, we have, by definition,

$$\mathbf{P}\Big\{E_n,\ \frac{S_n^{(n)}}{a_n} \geq g_2(1) - b\Big\} \ \geq\ \mathbf{P}\Big(\bigcap_{k=1}^N \Big\{g_1(\frac{i}{n}) \leq \frac{S_i^{(n)}}{a_n} \leq g_2(\frac{i}{n}),\ \forall i \in (m_{k-1},\, m_k] \cap \mathbb{Z},$$
$$g(\frac{m_k}{n}) \leq \frac{S_{m_k}^{(n)}}{a_n} \leq g(\frac{m_k}{n}) + 6\delta\Big\}\Big).$$

Applying the Markov property successively at times $m_{N-1},\ m_{N-2},\ \cdots,\ m_1$, we obtain, by writing $y_k := g(\frac{m_k}{n})$ for $1 \leq k \leq N$,

$$\mathbf{P}\Big\{E_n,\ \frac{S_n^{(n)}}{a_n} \geq g_2(1) - b\Big\} \geq p_{1,n}(0) \times \prod_{k=2}^N \inf_{y \in [y_{k-1},\, y_{k-1}+6\delta]} p_{k,n}(y),$$

where[7] for $1 \leq k \leq N$ and $y \in \mathbb{R}$,

$$p_{k,n}(y) \ :=\ \mathbf{P}\Big\{\alpha_{i,k,n} \leq \frac{S_i^{(n)}}{a_n} + y \leq \beta_{i,k,n},\ \forall i \leq \Delta_k;\ y_k \leq \frac{S_{\Delta_k}^{(n)}}{a_n} + y \leq y_k + 6\delta\Big\},$$
$$\alpha_{i,k,n} \ :=\ g_1(\frac{i+m_{k-1}}{n}), \qquad \beta_{i,k,n} := g_2(\frac{i+m_{k-1}}{n}), \qquad \Delta_k := m_k - m_{k-1}.$$

Uniform continuity of g guarantees that when n is sufficiently large, $|y_k - y_{k-1}| \leq \delta$ (for all $1 \leq k \leq N$, with $y_0 := 0$). In the rest of the proof, we will always assume that n is sufficiently large, say $n \geq n_0$, with n_0 depending on A and δ.

We need to bound $p_{1,n}(0) \times \prod_{k=2}^N \inf_{y \in [y_{k-1},\, y_{k-1}+6\delta]} p_{k,n}(y)$ from below. Let us first get rid of the infimum $\inf_{y \in [y_{k-1},\, y_{k-1}+6\delta]}$, which is the minimum between $\inf_{y \in [y_{k-1},\, y_{k-1}+3\delta]}$ and $\inf_{y \in [y_{k-1}+3\delta,\, y_{k-1}+6\delta]}$:

$$\inf_{y \in [y_{k-1},\, y_{k-1}+6\delta]} p_{k,n}(y) \geq \min\{p_{k,n}^{(1)},\, p_{k,n}^{(2)}\}, \qquad n \geq n_0,\ 2 \leq k \leq N,$$

where, for $1 \leq k \leq N$,

$$p_{k,n}^{(1)} \ :=\ \mathbf{P}\Big\{\alpha_{i,k,n} - y_{k-1} \leq \frac{S_i^{(n)}}{a_n} \leq \beta_{i,k,n} - y_{k-1} - 3\delta,\ \forall i \leq \Delta_k;\ \delta \leq \frac{S_{\Delta_k}^{(n)}}{a_n} \leq 2\delta\Big\},$$
$$p_{k,n}^{(2)} \ :=\ \mathbf{P}\Big\{\alpha_{i,k,n} - y_{k-1} - 3\delta \leq \frac{S_i^{(n)}}{a_n} \leq \beta_{i,k,n} - y_{k-1} - 6\delta,\ \forall i \leq \Delta_k;\ -2\delta \leq \frac{S_{\Delta_k}^{(n)}}{a_n} \leq -\delta\Big\}.$$

And, of course, $p_{1,n}(0) \geq p_{1,n}^{(1)} \geq \min\{p_{1,n}^{(1)},\, p_{1,n}^{(2)}\}$. We arrive at the following estimate:

$$\mathbf{P}\Big\{E_n,\ \frac{S_n^{(n)}}{a_n} \geq g_2(1) - b\Big\} \geq \prod_{k=1}^N \min\{p_{k,n}^{(1)},\, p_{k,n}^{(2)}\} = \min\{p_{N,n}^{(1)},\, p_{N,n}^{(2)}\} \prod_{k=1}^{N-1} \min\{p_{k,n}^{(1)},\, p_{k,n}^{(2)}\}.$$

[7]For notational simplification, we write $\forall i \leq \Delta_k$ instead of $\forall i \in (0,\, \Delta_k] \cap \mathbb{Z}$.

First, we bound $p^{(1)}_{k,n}$ and $p^{(2)}_{k,n}$ from below, for $1 \le k \le N-1$ (in which case $\Delta_k = r_n$). We split the indices $k \in (0,\, N-1] \cap \mathbb{Z}$ into A blocs, by means of $(0,\, N-1] \cap \mathbb{Z} = \cup_{\ell=1}^{A} J_\ell$, where $J_\ell = J_\ell(n) := (\frac{(\ell-1)(N-1)}{A},\, \frac{\ell(N-1)}{A}] \cap \mathbb{Z}$. For indices k lying in a same bloc J_ℓ, we use a common lower bound for $\min\{p^{(1)}_{k,n},\, p^{(2)}_{k,n}\}$ as follows: assuming $k \in J_\ell$, we have, by (5.2), $\alpha_{i,k,n} \le g_1(\frac{\ell}{A}) + \delta$, $\beta_{i,k,n} \ge g_2(\frac{\ell}{A}) - \delta$ (for $n \ge n_0$ and $i \le r_n$), and $|y_{k-1} - g(\frac{\ell}{A})| \le \delta$, which leads to: $p^{(1)}_{k,n} \ge q^{(1)}_{\ell,n}$, and $p^{(2)}_{k,n} \ge q^{(2)}_{\ell,n}$ (for $n \ge n_0$, $1 \le \ell \le A$ and $k \in J_\ell$), where (recalling that $\Delta_k = r_n$ for $1 \le k \le N-1$)

$$
\begin{aligned}
q^{(1)}_{\ell,n} &:= \mathbf{P}\Big\{ g_1(\frac{\ell}{A}) - g(\frac{\ell}{A}) + 2\delta \le \frac{S^{(n)}_i}{a_n} \le g_2(\frac{\ell}{A}) - g(\frac{\ell}{A}) - 5\delta,\ \forall i \le r_n;\ \delta \le \frac{S^{(n)}_{r_n}}{a_n} \le 2\delta \Big\},\\
q^{(2)}_{\ell,n} &:= \mathbf{P}\Big\{ g_1(\frac{\ell}{A}) - g(\frac{\ell}{A}) - \delta \le \frac{S^{(n)}_i}{a_n} \le g_2(\frac{\ell}{A}) - g(\frac{\ell}{A}) - 8\delta,\ \forall i \le r_n;\ -2\delta \le \frac{S^{(n)}_{r_n}}{a_n} \le -\delta \Big\}.
\end{aligned}
$$

Therefore,

$$
\mathbf{P}\Big\{ E_n,\ \frac{S^{(n)}_n}{a_n} \ge g_2(1) - b \Big\} \ge \min\{p^{(1)}_{N,n},\, p^{(2)}_{N,n}\} \prod_{\ell=1}^{A} \Big(\min\{q^{(1)}_{\ell,n},\, q^{(2)}_{\ell,n}\} \Big)^{\# J_\ell}.
$$

Since $\# J_\ell \le \frac{N}{A} \le \frac{n}{r_n A} \le \frac{n}{(A a_n^2 - 1)A}$, this yields

$$
\mathbf{P}\Big\{ E_n,\ \frac{S^{(n)}_n}{a_n} \ge g_2(1) - b \Big\} \ge \min\{p^{(1)}_{N,n},\, p^{(2)}_{N,n}\} \prod_{\ell=1}^{A} \Big(\min\{q^{(1)}_{\ell,n},\, q^{(2)}_{\ell,n}\} \Big)^{n/[(A a_n^2 - 1)A]}. \tag{5.3}
$$

It is well-known that the linear interpolation function $t(\in [0,1]) \to \frac{1}{a_n}\{S^{(n)}_{\lfloor r_n t \rfloor} + (r_n t - \lfloor r_n t \rfloor) X^{(n)}_{\lfloor r_n t \rfloor + 1}\}$ converges in law to $(\sigma\sqrt{A}\, W_t, 0 \le t \le 1)$, where W denotes a standard one-dimensional Brownian motion.[8] So, if we write

$$
f(a,b,c,d) := \mathbf{P}\Big\{ a \le W_t \le b,\ \forall\, t \in [0,\, 1];\ c \le W_1 \le d \Big\},
$$

for $a < 0 < b$ and $a \le c < d \le b$, then for any $1 \le \ell \le A$,

$$
\begin{aligned}
\lim_{n\to\infty} q^{(1)}_{\ell,n} &= f\Big(\frac{g_1(\frac{\ell}{A}) - g(\frac{\ell}{A}) + 2\delta}{\sigma A^{1/2}},\ \frac{g_2(\frac{\ell}{A}) - g(\frac{\ell}{A}) - 5\delta}{\sigma A^{1/2}},\ \frac{\delta}{\sigma A^{1/2}},\ \frac{2\delta}{\sigma A^{1/2}} \Big),\\
\lim_{n\to\infty} q^{(2)}_{\ell,n} &= f\Big(\frac{g_1(\frac{\ell}{A}) - g(\frac{\ell}{A}) - \delta}{\sigma A^{1/2}},\ \frac{g_2(\frac{\ell}{A}) - g(\frac{\ell}{A}) - 8\delta}{\sigma A^{1/2}},\ -\frac{2\delta}{\sigma A^{1/2}},\ -\frac{\delta}{\sigma A^{1/2}} \Big).
\end{aligned}
$$

[Thanks to (5.1), the limits are (strictly) positive.] The function f is explicitly known (see for example, Itô and McKean [10], p. 31):

$$
f(a,b,c,d) = \int_c^d \frac{2}{b-a} \sum_{n=1}^{\infty} \exp\Big(-\frac{n^2\pi^2}{2(b-a)^2} \Big) \sin\Big(\frac{n\pi|a|}{b-a} \Big) \sin\Big(\frac{n\pi(z-a)}{b-a} \Big)\, \mathrm{d}z, \tag{5.4}
$$

[8] In fact, finite-dimensional convergence is easily obtained by verifying Lindeberg's condition in the central limit theorem, whereas tightness is checked using a standard argument, see for example Billingsley [4].

from which it is easily seen that for all A sufficiently large, say $A \geq A_0$ (A_0 depending on δ), uniformly in $1 \leq \ell \leq A$,

$$\begin{aligned}
\lim_{n\to\infty} q_{\ell,n}^{(1)} &\geq \exp\Big(-\frac{\sigma^2\pi^2}{2} \frac{(1+\delta)A}{[g_2(\frac{\ell}{A}) - g_1(\frac{\ell}{A}) - 7\delta]^2}\Big), \\
\lim_{n\to\infty} q_{\ell,n}^{(2)} &\geq \exp\Big(-\frac{\sigma^2\pi^2}{2} \frac{(1+\delta)A}{[g_2(\frac{\ell}{A}) - g_1(\frac{\ell}{A}) - 7\delta]^2}\Big).
\end{aligned}$$

Similarly, we have a lower bound for $\min\{p_{N,n}^{(1)}, p_{N,n}^{(2)}\}$, the only difference being that Δ_N is not exactly r_n but lies somewhere between r_n and $2r_n$. This time, we only need a rough estimate: there exists a constant $C > 0$ such that

$$\liminf_{n\to\infty} \min\{p_{N,n}^{(1)}, p_{N,n}^{(2)}\} \geq C.$$

In view of (5.3), we get that, for all A sufficiently large (how large depending on δ),

$$\begin{aligned}
\liminf_{n\to\infty} \frac{a_n^2}{n} \log \mathbf{P}\Big\{E_n,\ \frac{S_n^{(n)}}{a_n} \geq g_2(1) - b\Big\} &\geq -\frac{\sigma^2\pi^2}{2}\frac{1}{A}\sum_{\ell=1}^{A} \frac{1+\delta}{[g_2(\frac{\ell}{A}) - g_1(\frac{\ell}{A}) - 7\delta]^2} \\
&\geq -\frac{\sigma^2\pi^2}{2}(1+2\delta)\int_0^1 \frac{\mathrm{d}t}{[g_2(t) - g_1(t) - 7\delta]^2}.
\end{aligned}$$

Letting $A \to \infty$ and $\delta \to 0$ (in this order), we obtain the desired lower bound in (2.13). $\square$

Upper bound in (2.12). The upper bound in (2.12) is needed in this paper only in the form of the original result of Mogulskii [18] (i.e., for sequences, instead of arrays, of random variables). We include its proof for the sake of completeness. It is similar to, and easier than, the proof of the lower bound in (2.13).

Let g be as before. Let $\delta > 0$ and $A > 0$ satisfy again (5.1) and (5.2), respectively. Let again $r_n := \lfloor A a_n^2 \rfloor$, $N = N(n) := \lfloor \frac{n}{r_n} \rfloor$. Let $m_k := k r_n$ for $0 \leq k \leq N-1$, but we are not interested in m_N any more. Write again $\alpha_{i,k,n} := g_1(\frac{i+m_{k-1}}{n})$ and $\beta_{i,k,n} := g_2(\frac{i+m_{k-1}}{n})$.

By the Markov property,

$$\mathbf{P}(E_n) \leq \prod_{k=2}^{N-1} \sup_{y\in[g_1(\frac{m_{k-1}}{n}),\, g_2(\frac{m_{k-1}}{n})]} \widetilde{p}_{k,n}(y),$$

where

$$\widetilde{p}_{k,n}(y) := \mathbf{P}\Big\{\alpha_{i,k,n} \leq \frac{S_i^{(n)}}{a_n} + y \leq \beta_{i,k,n},\ \forall 0 < i \leq r_n\Big\}.$$

Since g_1 and g_2 are bounded, we know that $g_1(\frac{m_{k-1}}{n})$ and $g_2(\frac{m_{k-1}}{n})$ lie in a compact interval, say $[-K\delta, K\delta]$ (K being an integer depending on δ). Therefore

$$\sup_{y\in[g_1(\frac{m_{k-1}}{n}),\, g_2(\frac{m_{k-1}}{n})]} \widetilde{p}_{k,n}(y) \le \max_{j\in[-K,\, K-1]\cap\mathbb{Z}} \sup_{y\in[j\delta,\,(j+1)\delta]} \widetilde{p}_{k,n}(y).$$

As in the proof of the lower bound in (2.13), we cut the interval $(1, N-1]\cap\mathbb{Z}$ into A blocs, by means of $(1, N-1]\cap\mathbb{Z} = \cup_{\ell=1}^{A} J_\ell$, where $J_\ell = J_\ell(n) := (\frac{(\ell-1)(N-2)}{A}+1,\, \frac{\ell(N-2)}{A}+1]\cap\mathbb{Z}$. For $k\in J_\ell$, we have, by (5.2), $\alpha_{i,k,n} \ge g_1(\frac{\ell}{A})-\delta$ and $\beta_{i,k,n} \le g_2(\frac{\ell}{A})+\delta$, which leads to: $\sup_{y\in[j\delta,\,(j+1)\delta]} \widetilde{p}_{k,n}(y) \le \widetilde{q}_{\ell,n}(j)$, where

$$\widetilde{q}_{\ell,n}(j) := \mathbf{P}\Big\{ g_1(\frac{\ell}{A}) - (j+2)\delta \le \frac{S_i^{(n)}}{a_n} \le g_2(\frac{\ell}{A}) - (j-1)\delta,\ \forall i \le r_n \Big\}.$$

Therefore,

$$\mathbf{P}(E_n) \le \prod_{\ell=1}^{A} [\max_{j\in[-K,\, K)\cap\mathbb{Z}} \widetilde{q}_{\ell,n}(j)]^{\# J_\ell}.$$

We have $\# J_\ell \ge \frac{N-2}{A} - 1 \ge \frac{n}{A^2 a_n^2} - \frac{3}{A} - 1$. On the other hand, for each pair (ℓ, j), $\widetilde{q}_{\ell,n}(j)$ converges (as $n\to\infty$) to $\mathbf{P}\{g_1(\frac{\ell}{A}) - (j+2)\delta \le \sigma A^{1/2} W_t \le g_1(\frac{\ell}{A}) - (j-1)\delta,\ \forall t\in[0,1]\}$, which, in view of (5.4), is bounded by $\exp\{-\frac{\pi^2\sigma^2}{2}\frac{(1-\delta)A}{[g_2(\ell/A)-g_1(\ell/A)-3\delta]^2}\}$ for all sufficiently large A and uniformly in (ℓ, j). Accordingly,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{a_n^2}{n} \log \mathbf{P}(E_n) &\le -\frac{\pi^2\sigma^2}{2}\frac{1}{A}\sum_{\ell=1}^{A} \frac{1-\delta}{[g_2(\frac{\ell}{A}) - g_1(\frac{\ell}{A}) - 3\delta]^2} \\
&\le -\frac{\pi^2\sigma^2}{2}(1-2\delta)\int_0^1 \frac{\mathrm{d}t}{[g_2(t)-g_1(t)-3\delta]^2},
\end{aligned}$$

for all sufficiently large A. Since δ can be as close to 0 as possible, this yields the upper bound in (2.12). $\square$

Acknowledgements

We are grateful to Jean Bérard for pointing out two errors in the first draft of the manuscript.

References

[1] Aldous, D. J. (1998). A Metropolis-type optimization algorithm on the infinite tree. *Algorithmica* **22**, 388–412.

[2] Biggins, J. D. (1976). The first- and last-birth problems for a multitype age-dependent branching process. *Adv. Appl. Probab.* **8**, 446–459.

[3] Biggins, J. D. and Kyprianou, A. E. (2005). Fixed points of the smoothing transform: the boundary case. *Electron. J. Probab.* **10**, Paper no. 17, 609–631.

[4] Billingsley, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

[5] Brunet, É. and Derrida, B. (1997). Shift in the velocity of a front due to a cutoff. *Phys. Rev. E* **56**, no. 3, 2597–2604.

[6] Derrida, B. and Simon, D. (2007). The survival probability of a branching random walk in presence of an absorbing wall. *Europhys. Lett. EPL* **78**, no. 6, Paper no. 60006.

[7] Derrida, B. and Simon, D. (2008). Quasi-stationary regime of a branching random walk in presence of an absorbing wall. *J. Stat. Phys* **131**, no. 2, 203–233.

[8] Hammersley, J. M. (1974). Postulates for subadditive processes. *Ann. Probab.* **2**, 652–680.

[9] Hu, Y. and Shi, Z. (2009). Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. *Ann. Probab.* **37**, 742–789.

[10] Itô, K. and McKean, H. P. (1965). *Diffusion Processes and their Sample Paths.* Springer, Berlin.

[11] Jaffuel, B. (2009+). The critical barrier for the survival of the branching random walk with absorption. `ArXiv math.PR/0911.2227`

[12] Kahane, J.-P. and Peyrière, J. (1976). Sur certaines martingales de Mandelbrot. *Adv. Math.* **22**, 131–145.

[13] Kesten, H. (1978). Branching Brownian motion with absorption. *Stoch. Proc. Appl.* **7**, 9–47.

[14] Kingman, J. F. C. (1975). The first birth problem for an age-dependent branching process. *Ann. Probab.* **3**, 790–801.

[15] Lyons, R. (1997). A simple path to Biggins' martingale convergence for branching random walk. In: *Classical and Modern Branching Processes* (Eds.: K. B. Athreya and P. Jagers). *IMA Volumes in Mathematics and its Applications* **84**, 217–221. Springer, New York.

[16] Lyons, R., Pemantle, R. and Peres, Y. (1995). Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.* **23**, 1125–1138.

[17] McDiarmid, C. (1995). Minimal positions in a branching random walk. *Ann. Appl. Probab.* **5**, 128–139.

[18] Mogulskii, A. A. (1974). Small deviations in the space of trajectories. *Theory Probab. Appl.* **19**, 726–736.

[19] Pemantle, R. (2009). Search cost for a nearly optimal path in a binary tree. *Ann. Appl. Probab.* **19**, 1273–1291.

Nina Gantert
Fachbereich Mathematik und Informatik
Universität Münster
Einsteinstrasse 62
D-48149 Münster
Germany
gantert@math.uni-muenster.de

Yueyun Hu
Département de Mathématiques
Université Paris XIII
99 avenue J-B Clément
F-93430 Villetaneuse
France
yueyun@math.univ-paris13.fr

Zhan Shi
Laboratoire de Probabilités UMR 7599
Université Paris VI
4 place Jussieu
F-75252 Paris Cedex 05
France
zhan.shi@upmc.fr